Hyperbolicity-Preserving Well-Balanced Stochastic Galerkin Method for Shallow Water Equations

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Abstract

We study the stochastic Galerkin (SG) method for stochastic parameterized shallow water equations. Our work comprises the following aspects:

- A hyperbolicity-preserving stochastic Galerkin formulation for the shallow water equations using only the conserved variables.
- A sufficient condition to preserve the hyperbolicity, which is a stochastic variant of the deterministic positivity condition.
- A computationally tractable condition to guarantee the hyperbolicity.
- A central-upwind scheme that preserves both the hyperbolicity and the well-balanced property at discrete time levels.

Motivations

- Uncertainties can enter the shallow water system, for example, via the noisy measurement of the bottom.
- A SG formulation of shallow water equations is not necessarily hyperbolic.
- A non-well-balanced scheme may lead to spurious oscillations on relatively coarse grid.

Stochastic Parameterized Shallow Water System

$$\frac{\partial h}{\partial t} + \frac{\partial q^x}{\partial x} + \frac{\partial q^y}{\partial y} = 0,$$

$$\frac{\partial q^x}{\partial t} + \frac{\partial}{\partial x}\left(\frac{(q^x)^2}{h} + \frac{gh^2}{2}\right) + \frac{\partial}{\partial y}\left(\frac{q^x q^y}{h}\right) = -gh\frac{\partial B}{\partial x},$$

$$\frac{\partial q^y}{\partial t} + \frac{\partial}{\partial x}\left(\frac{q^x q^y}{h}\right) + \frac{\partial}{\partial y}\left(\frac{(q^y)^2}{h} + \frac{gh^2}{2}\right) = -gh\frac{\partial B}{\partial y}.$$

h is the water height, q_x and q_y are the x- and y-discharges, and B is the time-independent surface. All the variables are ξ-dependent random fields, e.g., $h = h(x, y, t, \xi)$.

Polynomial Chaos Expansion (PCE)

- An unknown random field $z(x, y, t, \xi)$ is represented in the L^2_ρ orthonormal basis $\{\phi_k\}_{k\in\mathbb{R}}$, where $\rho := \rho(\xi)$ is the density of the random parameter.
$$z(x, y, t, \xi) = \sum_{k=1}^{\infty} \hat{z}_k(x, y, t)\phi_k(\xi),$$
- K-term truncated PCE:
$$\Pi_\Lambda[z] := \sum_{k=1}^{K} \hat{z}_k(x, y, t)\phi_k(\xi),$$
where Λ is the index set for the (possibly multivariate) polynomials, the cardinality of Λ is K, and $\phi_1(\xi) = 1$.
- A K-term PCE approximation to product of two random fields a and b:
$$\Pi_\Lambda[a, b] := \Pi_\Lambda\left[\Pi_\Lambda[a]\ \Pi_\Lambda[b]\right].$$
- A K-term PCE approximation to the ratio of two random fields a and b:
$$\Pi^\dagger_\Lambda[b/a] : \text{the solution to } \Pi_\Lambda[a, b/a] = \Pi_\Lambda[b].$$

Notations

- $\hat{z} = (\hat{z}_1, \ldots, \hat{z}_K)^\top$.
- $\mathcal{P}(\hat{z}) := \sum_{k=1}^{K} \hat{z}_k \mathcal{M}_k, \quad (\mathcal{M}_k)_{\ell m} = \langle \phi_k, \phi_\ell \phi_m \rangle_\rho$.
- It can be shown that
$$\widehat{\Pi_\Lambda[a, b]} = \mathcal{P}(\hat{a})\hat{b}, \qquad \widehat{\Pi^\dagger_\Lambda[b/a]} = \mathcal{P}^{-1}(\hat{a})\hat{b}.$$

Stochastic Galerkin (SG) Method

- Ansatz:
$$h \simeq h_\Lambda := \sum_{k=1}^{K} \hat{h}_k(x, y, t)\phi_k(\xi),$$
$$q^x \simeq q^x_\Lambda := \sum_{k=1}^{K} (\widehat{q^x})_k(x, y, t)\phi_k(\xi),$$
$$q^y \simeq q^y_\Lambda := \sum_{k=1}^{K} (\widehat{q^y})_k(x, y, t)\phi_k(\xi),$$
- Stochastic Galerkin method applies standard Galerkin procedure in the stochastic ξ space, which leads to a new system of partial differential equations with respect to the PCE coefficients.

SG Projection of Nonlinear Terms

❶
$$\frac{(q^x)^2}{h} = \frac{q^x}{h}\, q^x \longrightarrow \Pi_\Lambda\left[\frac{(q^x_\Lambda)^2}{h_\Lambda}\right] = \Pi_\Lambda\left[q^x_\Lambda\, \Pi^\dagger_\Lambda\left[\frac{q^x_\Lambda}{h_\Lambda}\right]\right],$$
$$\frac{(q^y)^2}{h} = \frac{q^y}{h}\, q^y \longrightarrow \Pi_\Lambda\left[\frac{(q^y_\Lambda)^2}{h_\Lambda}\right] = \Pi_\Lambda\left[q^y_\Lambda\, \Pi^\dagger_\Lambda\left[\frac{q^y_\Lambda}{h_\Lambda}\right]\right].$$

❷ For $q^x q^y/h$ in $(q^x q^y/h)_x$,
$$\frac{q^x q^y}{h} = (q^x)\frac{q^y}{h} \longrightarrow \Pi_\Lambda\left[\frac{q^x_\Lambda q^y_\Lambda}{h_\Lambda}\right] = \Pi_\Lambda\left[q^x_\Lambda\, \Pi^\dagger_\Lambda\left[\frac{q^y_\Lambda}{h_\Lambda}\right]\right].$$

❸ For $q^x q^y/h$ in $(q^x q^y/h)_y$,
$$\frac{q^x q^y}{h} = (q^y)\frac{q^x}{h} \longrightarrow \Pi_\Lambda\left[\frac{q^x_\Lambda q^y_\Lambda}{h_\Lambda}\right] = \Pi_\Lambda\left[q^y_\Lambda\, \Pi^\dagger_\Lambda\left[\frac{q^x_\Lambda}{h_\Lambda}\right]\right].$$

The Main Results

Hyperbolicity-Preserving SG Formulation

$$\frac{\partial}{\partial t}(\widehat{U}) + \frac{\partial}{\partial x}(\widehat{F}(\widehat{U})) + \frac{\partial}{\partial y}(\widehat{G}(\widehat{U})) = \hat{S}(\widehat{U}, \widehat{B}). \tag{1}$$

Here, $\widehat{U} := (\hat{h}^\top, \widehat{q^x}^\top, \widehat{q^y}^\top)^\top$, and

$$\hat{F}(\hat{U}) = \begin{pmatrix} \widehat{q^x} \\ \mathcal{P}(\widehat{q^x})\mathcal{P}^{-1}(\hat{h})\widehat{q^x} + \frac{1}{2}g\mathcal{P}(\hat{h})\hat{h} \\ \mathcal{P}(\widehat{q^x})\mathcal{P}^{-1}(\hat{h})\widehat{q^y} \end{pmatrix}, \quad \hat{G}(\hat{U}) = \begin{pmatrix} \widehat{q^y} \\ \mathcal{P}(\widehat{q^y})\mathcal{P}^{-1}(\hat{h})\widehat{q^x} \\ \mathcal{P}(\widehat{q^y})\mathcal{P}^{-1}(\hat{h})\widehat{q^y} + \frac{1}{2}g\mathcal{P}(\hat{h})\hat{h} \end{pmatrix}, \quad \hat{S}(\hat{U}, \hat{B}) = \begin{pmatrix} 0 \\ -g\mathcal{P}(\hat{h})\widehat{B}_x \\ -g\mathcal{P}(\hat{h})\widehat{B}_y \end{pmatrix}. \tag{2}$$

Theorem (Hyperbolicity-preserving condition)

The system (1) is hyperbolic if the matrix $\mathcal{P}(\hat{h}) > 0$.

The condition $\mathcal{P}(\hat{h}) > 0$ reduces to $h > 0$ when the ξ-dependence is dropped from the system.

Theorem (A computationally tractable condition)

Given Λ, let nodes ξ_m and weights τ_m satisfying $\{(\xi_m, \tau_m)\}_{m=1}^{M}$ represent any M-point positive quadrature rule that is exact on

$$P^3_\Lambda := \operatorname{span}\left\{\prod_{n=1}^{3} \phi_n \;\middle|\; n \in [K]\right\}.$$

If

$$h_\Lambda(x, y, t, \xi_m) > 0 \;\; \forall\, m = 1, \ldots, M, \tag{3}$$

then the matrix $\mathcal{P}(\hat{h}) > 0$.

In other words, we only need to ensure the positivity of the stochastic water heights at some quadrature points to preserve the hyperbolicity of (1).

Second-Order Central-Upwind Scheme

Assuming uniform rectangular partition over a rectangular region,

$$\frac{d}{dt}\boldsymbol{U}_{i,j} = -\frac{\mathcal{F}_{i+\frac{1}{2},j} - \mathcal{F}_{i-\frac{1}{2},j}}{\Delta x} - \frac{\mathcal{G}_{i,j+\frac{1}{2}} - \mathcal{G}_{i,j-\frac{1}{2}}}{\Delta y} + \overline{\boldsymbol{S}}_{i,j},$$

where $\boldsymbol{U}_{i,j}$ represent the cell averages of the vector $\widehat{U}$ in rectangular cell $\mathcal{C}_{i,j}$.

- Source term:
$$\overline{\boldsymbol{S}}_{i,j} \approx \frac{1}{|\mathcal{C}_{i,j}|}\int_{\mathcal{C}_{i,j}} \hat{S}(\widehat{U}, \widehat{B})\,dxdy.$$
- Numerical fluxes:
$$\mathcal{F}_{i+\frac{1}{2},j} := \frac{a^+_{i+\frac{1}{2},j}\widehat{F}(\boldsymbol{U}^E_{i,j}) - a^-_{i+\frac{1}{2},j}\widehat{F}(\boldsymbol{U}^W_{i+1,j})}{a^+_{i+\frac{1}{2},j} - a^-_{i+\frac{1}{2},j}} + \frac{a^+_{i+\frac{1}{2},j}a^-_{i+\frac{1}{2},j}}{a^+_{i+\frac{1}{2},j} - a^-_{i+\frac{1}{2},j}}\left[\boldsymbol{U}^W_{i+1,j} - \boldsymbol{U}^E_{i,j}\right],$$
$$\mathcal{G}_{i+\frac{1}{2},j} := \frac{b^+_{i,j+\frac{1}{2}}\widehat{G}(\boldsymbol{U}^N_{i,j}) - b^-_{i,j+\frac{1}{2}}\widehat{G}(\boldsymbol{U}^S_{i,j+1})}{b^+_{i,j+\frac{1}{2}} - b^-_{i,j+\frac{1}{2}}} + \frac{b^+_{i,j+\frac{1}{2}}b^-_{i,j+\frac{1}{2}}}{b^+_{i,j+\frac{1}{2}} - b^-_{i,j+\frac{1}{2}}}\left[\boldsymbol{U}^S_{i,j+1} - \boldsymbol{U}^N_{i,j}\right].$$
- Propagation speeds:
$$a^-_{i+\frac{1}{2},j} = \min\left\{\lambda_1\left(\frac{\partial\widehat{F}}{\partial\widehat{U}}(\boldsymbol{U}^W_{i+1,j})\right), \lambda_1\left(\frac{\partial\widehat{F}}{\partial\widehat{U}}(\boldsymbol{U}^E_{i,j})\right), 0\right\},$$
$$a^+_{i+\frac{1}{2},j} = \max\left\{\lambda_{3K}\left(\frac{\partial\widehat{F}}{\partial\widehat{U}}(\boldsymbol{U}^W_{i+1,j})\right), \lambda_{3K}\left(\frac{\partial\widehat{F}}{\partial\widehat{U}}(\boldsymbol{U}^E_{i,j})\right), 0\right\},$$
$$b^-_{i,j+\frac{1}{2}} = \min\left\{\lambda_1\left(\frac{\partial\widehat{G}}{\partial\widehat{U}}(\boldsymbol{U}^S_{i,j+1})\right), \lambda_1\left(\frac{\partial\widehat{G}}{\partial\widehat{U}}(\boldsymbol{U}^N_{i,j})\right), 0\right\},$$
$$b^+_{i,j+\frac{1}{2}} = \max\left\{\lambda_{3K}\left(\frac{\partial\widehat{G}}{\partial\widehat{U}}(\boldsymbol{U}^S_{i,j+1})\right), \lambda_{3K}\left(\frac{\partial\widehat{G}}{\partial\widehat{U}}(\boldsymbol{U}^N_{i,j})\right), 0\right\},$$

$\boldsymbol{U}^{E,W,N,S}_{i,j}$ are the pointwise values of the second-order accurate, non-oscillatory piecewise linear reconstructions of $\boldsymbol{U}_{i,j}$ at the midpoints of the boundaries, i.e.,

$$\boldsymbol{U}^E_{i,j} = \boldsymbol{U}_{i,j} + \frac{\Delta x}{2}(\boldsymbol{U}_x)_{i,j}, \quad \boldsymbol{U}^W_{i,j} = \boldsymbol{U}_{i,j} - \frac{\Delta x}{2}(\boldsymbol{U}_x)_{i,j},$$
$$\boldsymbol{U}^N_{i,j} = \boldsymbol{U}_{i,j} + \frac{\Delta y}{2}(\boldsymbol{U}_y)_{i,j}, \quad \boldsymbol{U}^S_{i,j} = \boldsymbol{U}_{i,j} - \frac{\Delta y}{2}(\boldsymbol{U}_y)_{i,j},$$

Hyperbolicity-Preserving Well-Balanced Central-Upwind Scheme

- Stochastic "lake-at-rest" state:
$$\begin{cases} q^x_\Lambda = q^y_\Lambda \equiv 0, \\ h_\Lambda + \Pi_\Lambda[B] \equiv C(\xi), \end{cases} \Rightarrow \begin{cases} \widehat{q^x} = \widehat{q^y} \equiv \mathbf{0}, \\ \hat{h} + \widehat{B} \equiv \widehat{C}. \end{cases}$$
- The PCE vector $\widehat{B}$ for the bottom function is replaced by its piecewise bilinear interpolant.
- The pointwise values of the reconstructions of the PCE of water surface $\hat{\eta}$ are reconstructed. The reconstructed water height are computed by $\hat{h} := \hat{\eta} - \widehat{B}$.
- The first moments $\hat{h}_1$ are "corrected" following a similar procedure to the central-upwind scheme for the deterministic shallow water equations.
- The PCE vectors $\hat{h}$ are filtered to satisfies the condition (3).

Numerical simulations

Deterministic initial water surface:

$$\eta(x, y, 0, \xi) = \begin{cases} 1.01, & \text{if } 0.05 < x < 0.15, \\ 1, & \text{otherwise}, \end{cases}$$

Deterministic initial velocity field (0-discharge):

$$u(x, y, 0, \xi) = v(x, y, 0, \xi) = 0.$$

Stochastic bottom topography:

$$B(x, y, \xi) = 0.8e^{-5(x-0.9+0.1\xi^{(1)})^2 - 50(y-0.5+0.1\xi^{(2)})^2}.$$

Randomness:

$$\xi^{(1)} \sim \text{Beta}(4, 2), \qquad \xi^{(2)} \sim \mathcal{U}(-1, 1).$$

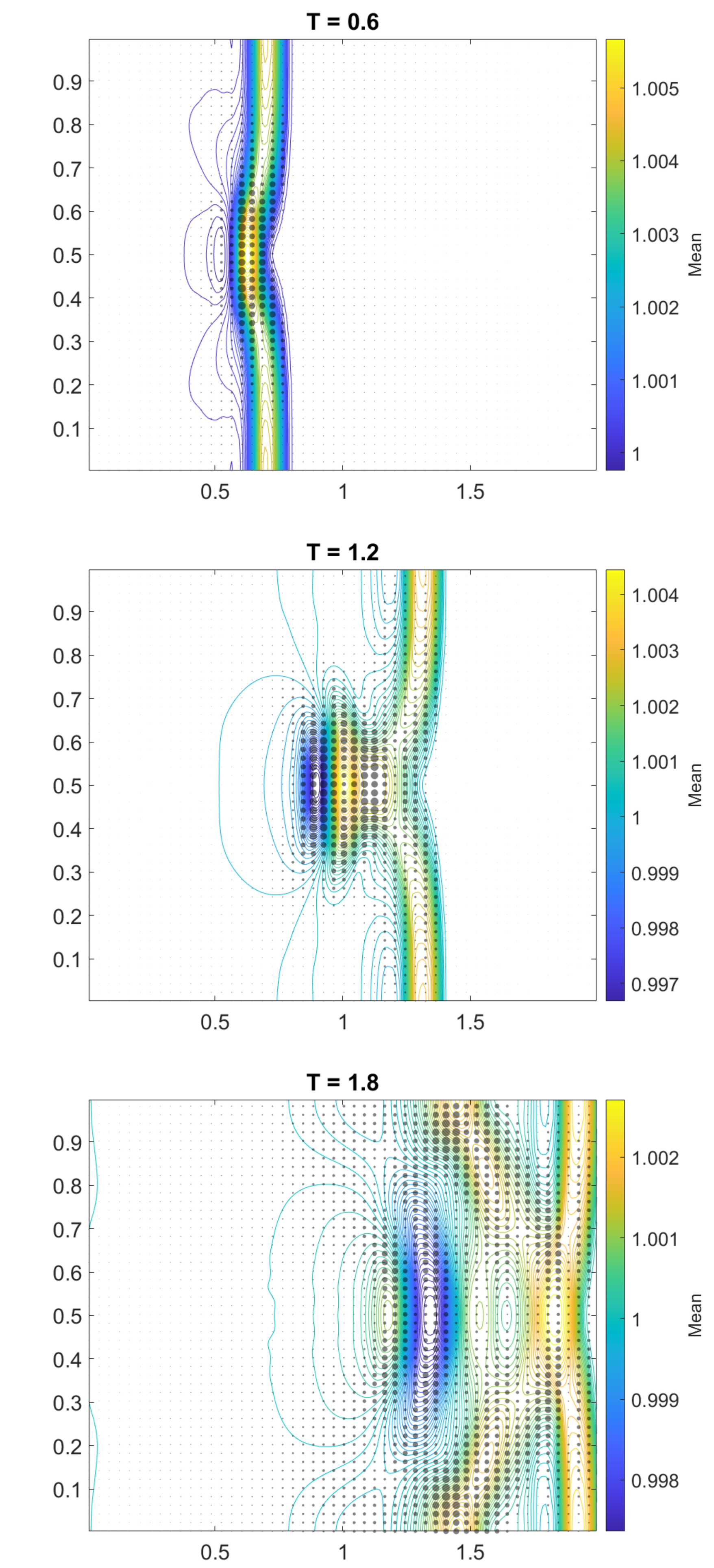

Figure 1:Numerical results at $T = 0.6$ (top), $T = 1.2$ (middle), and $T = 1.8$ (bottom), respectively. The largest disks are corresponding to the standard deviation values 2.20e-3, 2.00e-3, and 1.20e-3, respectively. The index set $\Lambda = \{(\nu^{(1)}, \nu^{(2)}) \in \mathbb{N}^2 \mid 0 \le \nu^{(1)}, \nu^{(2)} \le 3\}$. The polynomial basis is chosen to be the tensor-product set. Animation can be found in `https://ibit.ly/Q6H4`

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